A NOTE ON CRITICAL POINTS.

By F. H. Safford.

In the Annals of Mathematics, February, 1898, Dr. C. L. Bouton mentions, on page 26, the following function:

$$w=\frac{\displaystyle\sum_{k=0}^{k=p}\frac{(-1)^k z^{n-km}}{(p-k)\,!\,k\,!\,[n-km]\,[n-(k+1)\,m]\,\ldots\,[n-(k+p)\,m]}}{\displaystyle\sum_{k=0}^{k=p}\frac{(-1)^k z^{(p-k)m}}{(p-k)\,!\,k\,!\,[n-(p-k)\,m]\,[n-(p-k+1)\,m]\,\ldots\,[n-(2p-k)\,m]}}$$

where $1\leqq m\leqq\frac{n-1}{p}$, $m\neq\frac{n}{p+k}$ for $k=1,2\,\ldots\,p$; and $n\geqq 2\,p+1$; and says that it seems probable that the critical points are given by the equation:

$$z^{n-pm-1}\,(z^m-1)^{2p}=0.$$

To prove this surmise correct, multiply the numerator and denominator of w by the expression: $p\,!\,[n\,(n-m)\;\ldots\;(n-pm)]$ and write the denominator in reverse order.

Letting $z^m=x$, $\frac{n}{m}=s=-\alpha$, $p=-\beta$, $p-s+1=\gamma$, we get for $(-1)^p w$ the value

$$\frac{x^s+\frac{\alpha.\beta}{1.\gamma}x^{s-1}+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^{s-2}\ldots\frac{\alpha(\alpha+1)\cdots(\alpha+p-1).\beta\ldots(\beta+p-1)}{1.2\ldots p.\gamma\ldots(\gamma+p-1)}x^{s-p}}{1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2\ldots\frac{\alpha(\alpha+1)\ldots(\alpha+p-1).\beta\ldots(\beta+p-1)}{1.2.\ldots p.\gamma\ldots(\gamma+p-1)}x^p}$$

The denominator of this fraction is:

$$y_1=F\,(\alpha,\beta,\gamma,x)=F\,(-s,-p,p-s+1,x),$$

and its numerator is:

$$y_2=x^s\,F\left(\alpha,\beta,\gamma,\frac{1}{x}\right)=x^s\,F\left(-s,-p,p-s+1,\frac{1}{x}\right),$$
$$=x^{-\alpha}\,F\left(\alpha,\alpha-\gamma+1,\alpha-\beta+1,\frac{1}{x}\right).$$

For the critical points of w, as usual,

$$\frac{dw}{dz}=mz^{m-1}\frac{dw}{dx}=mz^{m-1}(-1)^p\,\left[y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}\right]/y_1^2=0.$$

From Forsyth's Differential Equations pp. 192, 193, the values of y_1 and y_2 are merely his solutions I and IX, respectively, of the differential equation satisfied by $F\,(\alpha,\,\beta,\,\gamma,\,x)$. But from p. 201, art. 128

$$y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}=Cx^{-\gamma}\,(1-x)^{\gamma-\alpha-\beta-1}.$$

Replacing x, α, β, γ, by their values in terms of z, m, n, p, and bringing down the factor z^{m-1}, we get, finally, omitting constant factors,

$$z^{n-mp-1}\,(z^m-1)^{2p}=0.$$

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NOTE ON THE FUNCTION SATISFYING THE FUNCTIONAL RELATION $\phi(u)\cdot\phi(v)=\phi(u+v)$.*

BY E. B. WILSON.

The exponential function a^x,

1) is defined for all values of the argument x,

2) is single-valued,

3) satisfies the functional relation $\phi(u)\cdot\phi(v)=\phi(u+v)$.

Conversely any function which satisfies conditions 1, 2, 3 and is *continuous*, is the exponential function $y=[\phi(1)]^x$.†

The fundamental property on which the proof rests is that if x_0 be any value whatsoever and x any rational number, $\phi(x\cdot x_0)=[\phi(x_0)]^x$. This shows, to speak geometrically, that values of the variable commensurable with any particular value x_0, give points lying on the curve $y=[\phi(x_0)]^x$. These points do not cover the curve completely, but they are everywhere dense. If the condition of continuity, which up to this point of the proof has not been used, be now added, the function is completely determined as $[\phi(1)]^x$.

Whether or not a *discontinuous* function may satisfy the relation $\phi(u)\cdot\phi(v)=\phi(u+v)$ has, so far as the author of this note is aware, never been decided. We shall however prove the

THEOREM: *If a function exists, which is defined for all values of the argument, is single-valued, satisfies the functional relation* $\phi(u)\cdot\phi(v)=\phi(u+v)$,

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* This paper was presented to the American Mathematical Society at its meeting of April 29, 1899.

† For proof *cf.* Tannery: *Functions d'une Variable* §80, p. 120.